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Triples in Matroid Circuits

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Watkins and Mesner characterized edge-triples of a graph which are not in any circuit, and Chakravarti and Robertson solved the dual problem of edge-triples not in a bond. Here we give a common generalization, solving the problem for all binary matroids. Our main result is that if e, f, g are elements of a 3-connected, internally 4-connected binary matroid, then there is a circuit containing e, f, g, unless either $\{e, f, g\}$ is a cocircuit, or the matroid is graphic and e, f, g are edges of the graph with a common vertex. The more general non-binary problem is still open.

1. INTRODUCTION

A well-known theorem of Whitney [14] asserts that any pair of elements of a connected matroid are in a circuit. However, there are matroids of arbitrarily high connectivity with triples of elements which are not in circuits—for example, in a graph, G if e, f, g are three edges with a common vertex, then no circuit of $\mathcal{M}(G)$ (the polygon matroid of G) contains e, f, g and yet $\mathcal{M}(G)$ can have arbitrarily high connectivity. It has been an open problem to characterize those triples which are not in circuits. In this paper we solve this problem for binary matroids, although the non-binary case remains open.

Several people (implicitly in Watkins and Mesner [10], and [3, 4, 5]) have shown that if e, f, g are edges of a 3-connected simple graph G, then there is a circuit of G containing e, f and g unless either e, f, g have a common vertex, or they form a bond. [A *bond* of a graph G is a minimal set of edges the deletion of which increases the number of connected components. Thus, the bonds of G are the circuits of $\mathcal{M}^*(G)$, the bond matroid of G.] In a considerably more difficult theorem, Chakravarti and Robertson [1] showed that three edges e, f, g of a 3-connected, cyclically 4-connected graph are in a bond unless the graph is planar and can be drawn in the plane with e, f, g on the boundary of the same region. Our result provides a common generalization of these two theorems.

Section 2 contains a discussion of connectivity, and an explanation of how to reduce the problem to the 3-connected, internally 4-connected case. Section 3 gives an explicit characterization in this case and section 4 is a brief discussion of the problem for non-binary matroids.

We shall assume familiarity with matroid theory—for an introduction, see Welsh [11]—but we here explain some notation and terminology. $E(M)$ denotes the set of elements of the matroid M, and M^* denotes its dual matroid. If $X \subseteq E(M)$, $M\backslash X$ denotes the matroid with elements set $E(M)-X$ and the induced independence structure, and M/X denotes $(M^*\backslash X)^*$. We abbreviate $M\backslash\{x\}$ by $M\backslash x$, etc. For $X \subseteq E(M)$, $r_M(X)$ denotes the rank of X in M, and $r_M(E(M))$ is abbreviated by $r(M)$. A *cycle* of a binary matroid M is a subset of $E(M)$ expressible as a disjoint union of circuits. F_7 denotes the Fano matroid, the matroid represented by the seven non-zero 3-tuples over $GF(2)$.

2. CONNECTIVITY

Let $k>0$ be an integer, and let (X_1, X_2) be a partition of $E(M)$ such that

$$r_M(X_1)+r_M(X_2) \leq r(M)+k-1.$$

(X_1, X_2) is called

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(a) a *Tutte k-separation if* $|X_1|, |X_2| \geq k$
(b) an *internal k-separation* if $|X_1|, |X_2| \geq k+1$
(c) a *vertical k-separation* if $r_M(X_1), r_M(X_2) < r(M)$.
We say that M is *Tutte* (respectively *internally, vertically*) *k-connected* if M has no Tutte (respectively internal, vertical) k-separation for any $k' < k$. (For a discussion of vertical connectivity, see Cunningham [2].) We shall usually abbreviate 'Tutte k-connected' by 'k-connected', and '2-connected' by 'connected'.

We shall require the following lemmas. Proofs of the first two are left to the reader.

(2.1) *If M is* 3*-connected and has at least four elements, then it has no circuits or cocircuits of cardinality* 1 *or* 2.

(2.2) *If* (X_1, X_2) *is a vertical k-separation of M and* $x \in X_2$ *is in the closure of* X_1, *then* $(X_1 \cup \{x\}, X_2 - \{x\})$ *is a vertical k-separation of M.*

(2.3) *If M is binary,* 3*-connected, and vertically* 4*-connected, then it is internally* 4*-connected.*

PROOF. Suppose that (X_1, X_2) is an internal k'-separation for some $k' < 4$. Then $k' = 3$, since M is 3-connected, and so $|X_1|, |X_2| \geq 4$. Now (X_1, X_2) is not a vertical 3-separation, and so we may assume that $r_M(X_2) = r(M)$. But

$$r_M(X_1) + r_M(X_2) \leq r(M) + 2$$

and so $r_M(X_1) \leq 2$. Thus some subset of X_1 is a circuit of cardinality 1 or 2, since $|X_1| \geq 4$ and M is binary. This contradicts (2.1).

We shall also require the following results of [7].

(2.4) *If M is connected and* (X_1, X_2) *is a Tutte* 2*-separation, then there are matroids* M_1, M_2 *with* $E(M_i) = X_i \cup \{z\} (i = 1, 2)$ (*where z is a new element*) *such that* $C \subseteq E(M)$ *is a circuit of M if and only if either it is a circuit of one of* M_1, M_2 *or* $(C \cap X_i) \cup \{z\}$ *is a circuit of* M_i *for* $i = 1, 2$. M_1, M_2 *are isomorphic to minors of M, and so are binary if M is binary.*

(2.5) *If M is binary and* 3*-connected, and* (X_1, X_2) *is a Tutte* 3*-separation, then there are binary matroids* M_1, M_2 *with* $E(M_i) = X_i \cup Z (i = 1, 2)$ (*where Z is a set of three new elements*) *such that* $C \subseteq E(M)$ *is a cycle of M if and only if there are cycles* C_i *of* $M_i (i = 1, 2)$ *such that* $C = C_1 \triangle C_2$, *and such that Z is a cycle of both* M_1 *and* M_2.
[For sets X, Y, $X \triangle Y$ denotes $(X - Y) \cup (Y - X)$.]

In these two results, M_1 and M_2 are called the *parts* of M corresponding to the 2- or 3-separation. It is easy to see that the parts corresponding to a 2-separation have fewer elements than M. This is not necessarily true for a 3-separation because equality can occur. However, we shall only apply (2.5) to vertical 3-separations, and it is easy to see that the parts in this case have smaller rank than M. We shall use (2.4) and (2.5) to decompose matroids into 3-connected, vertically 4-connected pieces, and these observations guarantee that our decompositions terminate, that 'cycling' does not occur.

Let e, f, g be distinct elements of a binary matroid M. We wish to study conditions for the truth of the following statement.

(C) *There is no circuit C of M with* $e, f, g \in C$.

Our conditions for (C) can be given explicitly in a 'structural' form, or given in an algorithmic way, but the most natural presentation is by means of a combination of the two. Thus we shall give a series of lemmas which reduce the problem to the case when M is 3-connected and internally 4-connected, and we give a structural characterization for this special case. In this section we explain the reductions.

First, we evidently have

(2.6) *Suppose that* (X_1, X_2) *is a Tutte* 1*-separation of M. If* $e, f, g \in X_1$, *then* (C) *is true for M if and only if it is true for* $M \backslash X_2$. *If* $e, f \in X_1$ *and* $g \in X_2$ *then* (C) *is true.*

This reduces the problem to the connected case. The next lemma similarly reduces it to the 3-connected case.

(2.7) *Suppose that M is connected and that* (X_1, X_2) *is a Tutte* 2*-separation of M. Let* M_1, M_2 *be the corresponding parts, with common element z. If* $e, f, g \in X_1$ *then* (C) *is true for M if and only if it is true for* M_1. *If* $e, f \in X_1$ *and* $g \in X_2$ *then* (C) *is true for M if and only if e, f, z are in no circuit of* M_1.

(The proof is easy, and is left to the reader.)

(2.8) *Suppose that M is* 3*-connected, and that* (X_1, X_2) *is a vertical* 3*-separation of M with* $e, f, g \in X_1$. *Let* M_1, M_2 *be the corresponding parts. Then* (C) *is true for M if and only if it is true for* M_1.

(Again , the proof is routine, using the methods of [7], and we omit it.)

We may thus assume that M is 3-connected and has no vertical 3-separation (X_1, X_2) with $e, f, g \in X_1$. If $|E(M)| \leq 3$ the problem is trivial: we assume then that $|E(M)| \geq 4$. By (2.1) all circuits of M have cardinality at least 3. If there is no circuit of M of cardinality 3 containing f and g, take a new element e', and let M' be the binary matroid with element set $E(M) \cup \{e'\}$ such that $M' \backslash e' = M$ and $\{e', f, g\}$ is a cycle.

(2.9) M' *is* 3*-connected and has no vertical* 3*-separation* (X_1, X_2) *with* $e, f, g \in X_1$. *Moreover,* (C) *is true for M if and only if it is true for* M'.

PROOF. Clearly (C) is true for M if and only if it is true for M', because no circuit of M includes $\{e, e', f, g\}$. Suppose that (X_1, X_2) is a Tutte k'-separation of M' with $k' \leq 2$, and $e' \in X_1$ say. Now $(X_1 - \{e'\}, X_2)$ is not a Tutte k'-separation of M, and so $|X_1| = k' \leq 2$, and X_1 is either a circuit or a cocircuit of M'. If $k' = 1$ then e' is a loop or coloop of M', which is impossible since $\{e, f, g\}$ is a cycle and $\{f, g\}$ is not a cycle. Thus $k' = 2$, and $X_1 = \{e', h\}$ say. Now h is not a coloop of $M' \backslash e' = M$, and so $\{e', h\}$ is a circuit of M'. But $\{e', f, g\}$ is a cycle of M', and so $h' \neq f, g$, and $\{f, g, h\}$ is a cycle of M, contrary to our hypothesis. Thus M' is 3-connected.

Suppose now that (X_1, X_2) is a vertical 3-separation of M' with $e, f, g \in X_1$. Now e' is spanned by $\{f, g\}$ in M' and so by (2.2), $(X_1 - \{e'\}, X_2 - \{e'\})$ is a vertical 3-separation of M, a contradiction. This completes the proof of (2.9).

By repeating this procedure for the pairs e, g and e, f our problem is reduced to the case when M is 3-connected, and has no vertical 3-separation (X_1, X_2) with $e, f, g \in X_1$, and every pair of e, f, g are in a circuit of cardinality 3. But in these circumstances we have

(2.10) *M is vertically* 4*-connected.*

PROOF. Suppose that (X_1, X_2) is a vertical k'-separation with $k' \leq 3$. Then neither X_1 nor X_2 includes $\{e, f, g\}$, and so without loss of generality we assume that $e, f \in X_1$, $g \in X_2$. Let e', f', g' be elements of M such that $\{e', f, g\}$, $\{e, f', g\}$, $\{e, f, g'\}$ are circuits of M. If

$e' \in X_1$ or $f' \in X_1$ then g is in the closure of X_1, and so $(X_1 \cup \{g\}, X_2 - \{g\})$ is a vertical 3-separation of M with $e, f, g \in X_1 \cup \{g\}$ (by (2.2)) contrary to our hypothesis. Thus $e', f' \in X_2$; but then e, f are in the closure of X_2, and so $(X_1 - \{e, f\}, X_2 \cup \{e, f\})$ is a vertical 3-separation, again contrary to our hypothesis.

It therefore suffices to solve our problem for 3-connected, vertically 4-connected binary matroids. But by (2.3), such a matroid is internally 4-connected. We shall obtain an explicit characterization for (C) for 3-connected, internally 4-connected binary matroids in the next section.

3. The 4-Connected Case

Let M be a binary matroid, and let $A_1, \ldots, A_7$ be disjoint subsets of $E(M)$. We say that $A_1, \ldots, A_7$ have the *F-property* in M if the following conditions are satisfied:

(F1) *A_1, A_2, A_3, A_4 and at least two of A_5, A_6, A_7 are non-empty.*
(F2) *The following sets are cycles of M: $\varnothing$, $A_1 \cup A_2 \cup A_3 \cup A_4$, $A_1 \cup A_2 \cup A_5 \cup A_6$, $A_3 \cup A_4 \cup A_5 \cup A_6$, $A_1 \cup A_3 \cup A_5 \cap A_7$, $A_2 \cup A_4 \cup A_5 \cup A_7$, $A_1 \cup A_4 \cup A_6 \cup A_7$, $A_2 \cup A_3 \cup A_6 \cup A_7$.*
(F3) *No other subset of $A_1 \cup \cdots \cup A_7$ is a cycle of M.*

It follows that the seven non-empty cycles of (F2) are in fact circuits of M.

We require the following four lemmas. Throughout, let $A_1, \ldots, A_7$ have the F-property in M, and let $Z = A_1 \cup \cdots \cup A_7$.

(3.1) *For any $z \in E(M) - Z$, $A_1, \ldots, A_7$ have the F-property in $M \backslash z$.*

(3.2) *For any $z \in E(M) - Z$, $A_1, \ldots, A_7$ have the F-property in M/z unless there is a circuit of C of M with $C \cap Z \neq \varnothing$, $C - Z = \{z\}$.*

(3.3) *If for some A_j there is a circuit C with $C \cap Z \subseteq A_j$, $C \cap Z \neq \varnothing$, $|C - Z| = 1$, define $A_i' = A_i (1 \leqslant i \leqslant 7, i \neq j)$, $A_j' = A_j \triangle C$; then $A_1', \ldots, A_7'$ have the F-property in M.*

The proofs of these are straightforward and are left to the reader. Our fourth lemma is a little more complicated.

(3.4) *Suppose that $x \in A_5$, $y \in A_6$, $z \in E(M) - Z$ and there is a circuit C of M with $C \cap Z \neq \varnothing$, $C - Z = \{z\}$. Then one of the following is true:*
(a) *there is a circuit of M included in $Z \cup \{z\}$ containing x, y and z*
(b) *there is a circuit C' of M with $C' - Z = \{z\}$, $C' \cap Z \neq \varnothing$, such that $C' \cap Z$ is a subset of one of A_5, A_6, A_7.*

Proof. We assume that (a) is false. Let

$$\mathscr{C} = \{X \subseteq Z : X \cup \{z\} \quad \text{is a cycle of } M\}.$$

Then $\mathscr{C} \neq \varnothing$, and $\varnothing \notin \mathscr{C}$. If $X \in \mathscr{C}$ then $X \cup \{z\}$ is a cycle and so is a disjoint union of circuits of M. At most one of them contains z, and so the others are subsets of Z; but by (F1), (F2), (F3), no two circuits included in Z are disjoint. Thus if $X \in \mathscr{C}$ and $X \cup \{z\}$ is not a circuit, then X includes one of the non-empty cycles of (F2), C_0 say, and $X - C_0 \in \mathscr{C}$, and $(X - C_0) \cup \{z\}$ is a circuit.

We also have that if $X \in \mathscr{C}$ and C_0 is any cycle of M included in Z then $X \triangle C_0 \in \mathscr{C}$. Because of this and the fact that $\mathscr{C} \neq \varnothing$, there exists $X_1 \in \mathscr{C}$ with $x, y \in X_1$. Since (a) is false, $X_1 \cup \{z\}$ is not a circuit, and so X_1 includes one of the circuits of (F2). Thus X_1 includes at least two of A_1, A_2, A_3, A_4. Put

$$X_2 = X_1 \triangle (A_1 \cup A_2 \cup A_3 \cup A_4);$$

and then $X_2 \in \mathscr{C}$, and X_2 is disjoint from at least two of A_1, A_2, A_3, A_4, and $x, y \in X_2$. Since (a) is false, X_2 includes one of the circuits of (F2), C_0 say. Then $X_2 - C_0 \in \mathscr{C}$, and $X_2 - C_0 \subseteq A_5$, A_6 or A_7, and $(X_2 - C_0) \cup \{z\}$ is a circuit. This completes the proof.

Now let e, f, g be distinct elements of M, and let $A_1, \ldots, A_7$ be disjoint subsets of $E(M)$. We say that $A_1, \ldots, A_7$ have the *F'-property* in M if at least two of A_5, A_6, A_7 have non-empty intersection with $\{e, f, g\}$. We say that $A_1, \ldots, A_7$ have the *F''-property* in M if

$$\{e, f, g\} \cap (A_1 \cup A_2 \cup A_3 \cup A_4) = \varnothing$$

and there is no Tutte 3-separation (X_1, X_2) of M with $\{e, f, g\} \subseteq X_1$, $A_1 \cup A_2 \cup A_3 \cup A_4 \subseteq X_2$. The following lemma follows by an easy rank argument—see [7, 9].

(3.5) *If $A_1, \ldots, A_7$ have the F''-property in M, and $z \in E(M) - Z$, and $z \neq e, f, g$, then $A_1, \ldots, A_7$ have the F''-property in at least one of $M \backslash z$, M/z.*

Now we can establish the main step in the argument, the following.

(3.6) *Let e, f, g be distinct elements of a binary matroid M, and let $A_1, \ldots, A_7$ be disjoint subsets of $E(M)$ with the F-, F'- and F''-properties. Then there is a circuit of M containing e, f and g.*

PROOF. We proceed by induction on $|E(M)|$, and for fixed M, we proceed by induction on $|(A_5 \cup A_6 \cup A_7)|$. Put $A_1 \cup \cdots \cup A_7 = Z$. We assume without loss of generality that $e \in A_5$, $f \in A_6$.
(1) *$\{e, f, g\}$ includes no circuits or cocircuits of M.*
This is implied by the F''-property.
(2) *We may assume that M has no loops and Z spans M.*
For no element of Z is a loop. If $z \in E(M) - Z$, $z \neq g$, and there is no circuit C with $C \cap Z \neq \varnothing$, $C - Z = \{z\}$ then by (3.1) and (3.2), $A_1, \ldots, A_7$ have the F-property in both $M \backslash z$ and M/z. They also clearly have the F'-property in both matroids, and the F''-property in at least one of them, by (3.5), and so e, f, g are in a circuit of one of $M \backslash z$, M/z, by our first induction. Hence e, f, g are in a circuit of M. Thus we may assume there is no such element z. If follows that M has no loops, and Z spans $E(M) - \{g\}$ and hence $E(M)$, by (1).
(3) *We may assume that no $x \in E(M) - Z$ distinct from g is a parallel element of M.*
For suppose that $x \in E(M) - Z$, $x \neq g$, and $\{x, y\}$ is a circuit for some element y. By (3.1), $A_1, \ldots, A_7$ have the F-property in $M \backslash x$, and clearly also the F'-property. Suppose that (X_1, X_2) is a Tutte 3-separation of $M \backslash x$ with $\{e, f, g\} \subseteq X_1$ and $A_1 \cup A_2 \cup A_3 \cup A_4 \subseteq X_2$. Let $y \in X_k$ say; and then X_k spans x in M, and so $(X_k \cup \{x\}, E(M) - (X_k \cup \{x\}))$ is a Tutte 3-separation of M contrary to the F''-property. Thus $A_1, \ldots, A_7$ have the F''-property in $M \backslash x$ and the result follows by induction.
(4) *We may assume that M has no parallel elements.*
For suppose that $\{x, y\}$ is a circuit. $\{x, y\} \subseteq Z$ is impossible, by (F2) and (F3); and we may assume that $\{x, y\} \subseteq E(M) - Z$, by (3). Thus one of x, y is in Z and the other is g; $x \in Z$, $y = g$ say. Then $x \neq e, f$, by (1). Choose $j (1 \leq j \leq 7)$ such that $x \in A_j$. Define $A'_i =$

$A_i(1 \leq i \leq 7,\ i \neq j)$, $A'_j = A_j \triangle \{x, g\}$; then by (3.3), $A'_1, \ldots, A'_7$ have the F-property in M. If $j = 1, 2, 3$ or 4 then by (F2) there is a circuit of M containing e, f and g. We may assume then that $j = 5, 6$ or 7, and so $A_1, \ldots, A_7$ have the F'- and F''-properties, and the result follows from the argument of (3).

If $g \in A_1 \cup A_2$ then by (F2) and (F3) there is a circuit of M containing e, f, g. Thus we may assume that $g \in (E(M) - Z) \cup A_7$.

Case 1 $g \in E(M) - Z$.

Now Z spans g, and so by (3.4) we may assume that there is a circuit C of M with $C \cap Z \neq \emptyset$, $C \cap Z \subseteq A_5$, A_6 or A_7, and $C - Z = \{g\}$. By (4), $|C| \geq 3$. Define $A'_i = A_i(1 \leq i \leq 7,\ i \neq j)$ and $A'_j = A_j \triangle C$, where $C - \{z\} \subseteq A_j$. Then

$$|(A'_5 \cup A'_6 \cup A'_7)| \leq |(A_5 \cup A_6 \cup A_7)|,$$

and moreover, $A_1, \ldots, A_7$ have the F-, F'-, and F''-properties, by (3.3). The result then follows by our second induction.

Case 2 $g \in A_7$.

In this case the symmetry between e, f and g is restored. By (1), $\{e, f, g\}$ contains no cocircuit of M, and so there is a circuit C for which $|C \cap \{e, f, g\}|$ is odd. For each $z \in C - Z$, let C_z be a circuit with $C_z - Z = \{z\}$ (this exists by (2)). Put

$$C_0 = C \triangle \mathop{\triangle}_{z \in C - Z} C_z.$$

Then $C_0 \subseteq Z$ and C_0 is a cycle, and so by (F2), C_0 has even intersection with $\{e, f, g\}$. Thus there exists $z \in C - Z$ such that C_z has odd intersection with $\{e, f, g\}$.

Suppose that there is a circuit C' of M with $C' \subseteq Z \cup \{z\}$ and $e, f, z \in C'$. If $g \notin C'$ then C' has even intersection with $\{e, f, g\}$, and so the cycle $C_z \triangle C'$ is a subset of Z with odd intersection with $\{e, f, g\}$, a contradiction. Thus $g \in C'$ and the theorem is true. We may therefore assume that there is no such circuit C'. By (3.4), there is a circuit C_1 with $C_1 \cap Z \neq \emptyset$, $C_1 - Z = \{z\}$, and $C_1 \cap Z \subseteq A_5$, A_6 or A_7. By (4), $|C_1| \geq 3$. Define $A'_i = A_i(1 \leq i \leq 7,\ i \neq j)$, $A'_j = A_j \triangle C_1$ where $C_1 \cap Z \subseteq A_j$. Then $A'_1, \ldots, A'_7$ have the F-, F'-, and F''-properties in M, and

$$|A'_5 \cup A'_6 \cup A'_7| < |A_5 \cup A_6 \cup A_7|.$$

The result follows from our second induction. This completes the proof.

Let N be the binary matroid with six elements a, b, c, d, e, f isomorphic to $\mathcal{M}(K_4)$, in which $\{a, c, e\}$, $\{a, d, f\}$, $\{b, c, f\}$, $\{b, d, e\}$ are circuits. We require the following.

(3.7) *Suppose that N is a minor of the binary matroid M. Then there exist disjoint subsets $A_1, \ldots, A_7$ of $E(M)$ with the F-property, such that $a \in A_1$, $b \in A_2$, $c \in A_3$, $d \in A_4$, $e \in A_5$, $f \in A_6$.*

Proof. Express N as $M \backslash X / Y$ with $|Y|$ minimum. The result is true for M if it is true for $M \backslash X$, and so we may assume that $X = \emptyset$, and $N = M/Y$, and for each $y \in Y$, N is not a minor of $M \backslash y$. For each $y \in Y$, let M_y be $M/(Y - \{y\})$. Then $M_y / y = N$, and $M_y \backslash y \neq N$, since N is not a minor of $M \backslash y$. Thus y is not a loop or coloop of M_y. If y is not a series element of M_y, then M_y is a rank 4 binary matroid with seven elements with no coloops or series elements. Hence M^*_y is a rank 3 binary matroid with seven elements with no loops or parallel elements; that is, M^*_y is isomorphic to the Fano matroid F_7. Hence $M_y \cong F^*_7$ and so $\{y, e, f\}$ is a cocircuit of M_y. We deduce that for all $y \in Y$, either $\{y, z\}$ is a cocircuit of M for some $z \in \{a, \ldots, f\}$, or $\{y, e, f\}$ is a cocircuit of M. Put

$$\begin{aligned} A_1 &= \{a\} \cup \{y \in Y : \{y, a\} \text{ is a cocircuit of } M\} \\ &\vdots \\ A_6 &= \{f\} \cup \{y \in Y : \{y, f\} \text{ is a cocircuit of } M\} \\ A_7 &= \{y \in Y : \{y, e, f\} \text{ is a cocircuit of } M\}. \end{aligned}$$

Then $A_1 \cup \cdots \cup A_7 = E(M)$, and $A_1, \ldots, A_7$ are pairwise disjoint, and it is straightforward to verify that $A_1, \ldots, A_7$ have the F-property, as required.

(3.8) *If N is a minor of the binary matroid M, and $g \in E(M) - E(N)$, and there is no Tutte 3-separation (X_1, X_2) of M with $\{e, f, g\} \subseteq X_1$ and $\{a, b, c, d\} \subseteq X_2$, then there is a circuit of M containing e, f, and g.*

Proof. Choose $A_1, \ldots, A_7$ with the F-property as in (3.7). If $g \in A_1 \cup \cdots \cup A_4$ then by (F2) there is a circuit of M containing e, f and g. We assume then that $g \notin A_1 \cup \cdots \cup A_4$. There is no Tutte 3-separation (X_1, X_2) of M with $\{e, f, g\} \subseteq X_1$, $A_1 \cup \cdots \cup A_4 \subseteq X_2$, because $a \in A_1, \ldots, d \in A_4$. Thus $A_1, \ldots, A_7$ have the F-, F'- and F''-properties. The result follows from (3.6).

Let e, f be distinct elements of a binary matroid M. We say that e, f are *non-adjacent* in M if M has a minor M' isomorphic to $\mathcal{M}(K_4)$ with $e, f \in E(M')$, such that no circuit of M' of cardinality 3 contains both e and f; and e, f are *adjacent* otherwise.

(3.9) *Let M be a 3-connected, internally 4-connected binary matroid. Then either $\{e, f, g\}$ is a cocircuit, or there is a circuit of M containing e, f and g, or every pair of e, f, g are adjacent in M.*

Proof. Suppose that e, f say are not adjacent in M. Then there are elements a, b, c, d of M such that M has N (as defined earlier) as a minor. If $g = a$, b, c or d then there is a circuit of N (and hence a circuit of M) containing e, f, and g. We assume then that $g \neq a, b, c, d$. If there is a Tutte 3-separation (X_1, X_2) with $\{e, f, g\} \subseteq X_1$, $\{a, b, c, d\} \subseteq X_2$ then $|X_1| = 3$, since (X_1, X_2) is not a internal 3-separation, and so $X_1 = \{e, f, g\}$. Hence $\{e, f, g\}$ includes a circuit or cocircuit of M. No proper subset of $\{e, f, g\}$ is a circuit or cocircuit, because M is 3-connected. Thus either $\{e, f, g\}$ is a circuit or it is a cocircuit, and in either case the theorem is true. We assume therefore that there is no such Tutte 3-separation (X_1, X_2). The result follows from (3.8).

We require the following lemma, proved in [6].

(3.10) *Let M be a 3-connected, internally 4-connected binary matroid, and let e, f be distinct elements. Then e, f are adjacent in M if and only if there is a graph G with $\mathcal{M}(G) = M$ or M^* such that e, f are edges of G with a common vertex.*

Our main theorem is the following.

(3.11) *Let e, f, g be distinct elements of a 3-connected, internally 4-connected binary matroid. Then there is no circuit of M containing e, f and g, if and only if either $\{e, f, g\}$ is a cocircuit, or there is a graph G with $M = \mathcal{M}(G)$ such that e, f, g are edges of G with a common vertex.*

PROOF. The 'if' part is clear, and we prove 'only if'. Suppose then that there is no circuit of M containing e, f, g, and $\{e, f, g\}$ is not a cocircuit. By (3.9), every pair of e, f, g are adjacent in M, and so by (3.10) there are 3-connected simple graphs G_1, G_2, G_3 with

$$E(G_1) = E(G_2) = E(G_3) = E(M)$$

such that f, g have a common vertex in G_1, e, g have a common vertex in G_2, e, f have a common vertex in G_3, and for $i = 1, 2, 3$, $\mathcal{M}(G_i) = M$ or M^*. Some two of G_1, G_2, G_3 have the same polygon matroid, G_1, G_2 say; and so by a theorem of Whitney [12], G_1 and G_2 are isomorphic and e, g have a common vertex in G_1. If e, f have a common vertex in G_1, then either $\{e, f, g\}$ is a circuit of $\mathcal{M}(G_1)$, and so either is a circuit or cocircuit of M, contrary to our assumption, or e, f, g are all incident with some vertex v. In this case if $M = \mathcal{M}(G_1)$ the theorem is true, and if $M^* = \mathcal{M}(G_1)$ then there is a bond of G_1 (and hence a circuit of M) containing e, f and g, a contradiction. Thus we may assume that e, f have no common vertex in G_1. They do have a common vertex in G_3, and G_1, G_3 are 3-connected, and so by Whitney's theorem [12] $\mathcal{M}(G_1) \neq \mathcal{M}(G_3)$. Thus one of $\mathcal{M}(G_1)$, $\mathcal{M}(G_3)$ is M and the other is M^*. Hence both $\mathcal{M}(G_1)$ and $\mathcal{M}^*(G_1)$ are graphic, and so G_1 is planar, by a theorem of Whitney [13]. Take a drawing of G_1 in the plane, and let G be its geometric dual. Identify the edges of G with those of G_1 in the natural way. Then $\mathcal{M}(G) = \mathcal{M}^*(G_1) = \mathcal{M}(G_3)$, and G, G_3 are 3-connected; so by Whitney's theorem [12], e, f are adjacent in G. Thus e, f both lie on the boundary of some region R in our drawing of G_1. But G_1 is 3-connected, and g shares one end with e and the other with f. Thus g also lies on the boundary of R, and so e, f, g have a common vertex in G. There is a circuit of $\mathcal{M}^*(G)$ containing e, f and g, and so $\mathcal{M}^*(G) \neq M$. Thus $\mathcal{M}(G_1) \neq M$, and so $\mathcal{M}(G_1) = M^*$, that is, $\mathcal{M}(G) = M$. Then G satisfies the theorem.

4. The Non-Binary Case

The problem of finding a characterization for the truth of (C) for non-binary matroids is still unsolved. It presents several new difficulties:

(a) it is no longer sufficient for (C) that $\{e, f, g\}$ be a cocircuit

(b) there is a problem working with Tutte 3-separations of non-binary matroids in order to argue as in (2.8)

(c) there are 3-connected, vertically 4-connected, internally 4-connected non-binary matroids with triples which are not in circuits—for example, the non-Fano matroid (represented over the real field by the seven non-zero 0–1 3-tuples) can be extended to such a matroid, using the method of (2.9) and (2.10).

Nevertheless, I conjecture that essentially the same result is true, that all sufficiently highly connected counterexamples are graphic—see [8] for some related results.

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